Relationship between oscillatory thermal instability and dynamical thin-shell overstability of radiative shocks

J. Martin Laming*
Space Science Division Code 7674L, Naval Research Laboratory, Washington, DC 20375, USA
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We give an analytic treatment of radiative cooling behind radiative shocks following solutions given by Chevalier and Imamura. We demonstrate that within the approximation of a steady state radiative shock, the radiative cooling laws $\Lambda \propto T^{\alpha}$ that give rise to the oscillatory instability modeled by Chevalier and Imamura in $\gamma=5/3$ cooling gas are stable to the dynamical thin-shell overstability in this gas, and vice versa. We also show that the fundamental features of the dynamical overstability observed by Grun *et al.* can also be understood on these bases.

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I. INTRODUCTION

Radiative shock waves, loosely defined as shock waves with radiative cooling times for the shocked plasma shorter than the hydrodynamic evolutionary time scale, exhibit various modes of instability and overstability. For appropriate forms of the temperature dependence of the radiative cooling function, a radiative cooling zone exists some distance behind the shock front. Here the shocked plasma undergoes a thermal instability and cools catastrophically. The cooling rate is dependent on the shock velocity, and oscillations in the shock velocity and distance behind the shock front to the cooling zone may develop as the shock decelerates. This has been modeled for blast waves in the interstellar medium [1–3] as well as accretion column shocks in a magnetic white dwarf [4].

In a related phenomenon, a decelerating shock may become subject to further oscillations. Corrugations or ripples that grow as a power law of time may develop [5–9]. These ripples grow because the force due to the thermal pressure of the shocked gas, which is perpendicular to the local shock front, is not necessarily parallel to the force from the ram pressure of the upstream plasma, which is directed along the shock velocity vector. In shocks with sufficiently high compression this imbalance of forces induces oscillatory movement of material within the shock shell. Parts of the shell that contain less mass slow down more than the parts of the shell that contain more mass and a growing oscillation ensues. In its nonlinear phase [9] knots or clumps of material may form with sizes similar to the shocked shell thickness.

The existence of growing ripples in radiative shock fronts was demonstrated in a laboratory experiment by Grun *et al.* [10]. These authors produced blast waves in nitrogen and xenon gas and showed that shocks in the more radiative xenon gas rippled with a power-law growth rate similar to theoretical predictions (but still with significant discrepancies), whereas shocks in nitrogen remained stable. More recently other researchers, working in a somewhat different parameter space, attempted to produce the rippling overstability, but were unable to do so [11]. Recent papers [12,13] have pointed the way towards a more quantitative understanding of these phenomena. Detailed calculations of radiative cooling for Xe and N demonstrated that the shock compressions produced in N blast waves and in Xe blast waves with velocity below about 25 km s^{-1} were not sufficient to produce the overstability, and that the observations in Refs. [10,11] could be understood on these grounds. In these calculations, the mass swept up behind the blast wave and the corresponding overstability are generally dominated by gas that is in the process of cooling radiatively following shock passage, rather than the quasi-inert shell of cold gas generally treated in theories, e.g., Refs. [5–9]. This is also the gas responsible for the oscillatory thermal instability discussed briefly above, but the precise relationship between the two forms of instability/overstability in the experiment of Ref. [10] remains obscure. It is the aim of this Brief Report to elucidate this using an analytic approach which builds on the original work of Ref. [4]. Section II describes the analytic model, Sec. III discusses comparisons with experimental data, and Sec. IV concludes.

II. FORMALISM

We follow Ref. [4] which gives analytic solutions for the system of flow equations

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial v}{\partial x} + v\frac{\partial \rho}{\partial x} = 0, \tag{1}$$

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) + \frac{\partial p}{\partial x} = 0, \tag{2}$$

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} - \gamma\frac{p}{\rho}\left(\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x}\right) = -(\gamma-1)\rho^2 A\left(\frac{p}{\rho}\right)^{\alpha}, \tag{3}$$

for plasma pressure p, density ρ, velocity v, and polytropic index γ. The spatial coordinate is $x=0$ at the radiative cooling zone, and $x=x_s$ at the shock front. The radiative power losses are given by $\Lambda=A\rho^2(p/\rho)^{\alpha}$ where A is a constant. For boundary conditions $\rho(x_s)=\rho_{in}(\gamma+1)/(\gamma-1)$, $v(x_s)=-u_{in}(\gamma-1)/(\gamma+1)$, and $p(x_s)=2\rho_{in}u_{in}^2/(\gamma+1)$, analytic solutions to Eq. (1)–(3) can be found for various values of α, which are given in Table I in terms of the variables $\xi=x/x_s$ and $w=v/u_{in}$. Concentrating on the $\gamma=5/3$ case, Ref. [4] proceeded to perturb the position of the shock front, and solved numerically the system of six coupled first order differential equations that describe the (complex) perturbations to p, ρ,

*Electronic address: jlaming@ssd5.nrl.navy.mil

TABLE I. CI hydrodynamic solutions.

α	ξ
-1	$\left[\frac{\gamma+1}{6}w^6-\frac{2\gamma+1}{5}w^5+\frac{\gamma}{4}w^4\right]\left[\frac{(\gamma-1)^6}{(\gamma+1)^5}-\frac{(2\gamma+1)(\gamma-1)^5}{5(\gamma+1)^5}+\frac{\gamma(\gamma-1)^4}{4(\gamma+1)^4}\right]^{-1}$
0	$\left[\frac{\gamma+1}{4}w^4+\frac{\gamma}{3}w^3\right]\left(\frac{\gamma+1}{\gamma-1}\right)^3\frac{12}{3+\gamma}$
$1/2$	$\left[\sqrt{-w-w^2}\left\{-\frac{\gamma+1}{3}w^2+\frac{5-\gamma}{12}w+\frac{\gamma-5}{8}\right\}+\frac{5-\gamma}{16}\cos^{-1}(1+2w)\right]\left\{\sqrt{\frac{\gamma-1}{\gamma+1}-\left(\frac{\gamma-1}{\gamma+1}\right)^2}\right.$ $\left.\times\left[-\frac{\gamma+1}{3}\left(\frac{\gamma-1}{\gamma+1}\right)^2-\frac{5-\gamma}{12}\frac{\gamma-1}{\gamma+1}+\frac{\gamma-5}{8}+\frac{5-\gamma}{16}\cos^{-1}\left(1-2\frac{\gamma-1}{\gamma+1}\right)\right]\right\}^{-1}$
1	$\left[\frac{\gamma+1}{2}w^2-w+\ln(w+1)\right]\left[\frac{\gamma-1}{2}+\ln\frac{2}{\gamma+1}\right]^{-1}$
2	$\left[(\gamma+1)\ln(w+1)-\frac{w}{w+1}\right]\left[\frac{\gamma-1}{2}+(\gamma+1)\ln\frac{2}{\gamma+1}\right]^{-1}$

and v. They found the oscillatory thermal instability at the fundamental frequency (essentially u_{in}/x_s) for $\alpha<0.4$ and instability at the first and second harmonics for $\alpha<0.8$. The shock is stable for higher values of α.

We take advantage of the analytic solutions for the steady state to calculate the mass accreted behind the shock in the steady state for various values of α, given by

$$\int_0^{x_s}\rho dx = \rho_{in}x_s\frac{\int_{-(\gamma-1)/(\gamma+1)}^{w_f}(-w-w^2)^{-\alpha}[(\gamma+1)w^2+\gamma w]dw}{\int_{-(\gamma-1)/(\gamma+1)}^{w_f}(-w-w^2)^{-\alpha}[(\gamma+1)w^3+\gamma w^2]dw}, \tag{4}$$

integrating w from its initial value of $-(\gamma-1)/(\gamma+1)$ to its final value w_f. For $\alpha<2$ this final value of w can be taken as $w_f=0$ without problem, and a finite result for the postshock mass results. For $\alpha\geqslant 2$, the postshock mass diverges as $w_f\rightarrow 0$. We make the assumption that the heated postshock gas cools down to its initial preshock temperature, so $w_f=-\rho_{in}/\rho=-(T_{in}/T_s)(\gamma-1)/(\gamma+1)\simeq-(\gamma+1)/(2\gamma M^2)$ where M is the shock Mach number. Our solutions for the mean postshock density integrated over the distance from the shock front to the cooling zone in units of the postshock density immediately behind the shock front are given in Table II. In Table III we give the mean shock compressions for the different values of α, for $\gamma=5/3$, 4/3, and 7/5, corresponding to monatomic nonrelativistic and relativistic gas and diatomic molecular (nonrelativistic) gas, respectively.

We have also calculated the shock compressions and effective polytropic indices numerically for a wider range of α, which are plotted in Fig. 1. In Ref. [5] it was shown that for $\gamma_{eff}<1.2$ shocks may become overstable, depending on the Mach number, with overstability becoming possible at higher γ_{eff} for lower Mach number. Thus in the range of α for which the oscillatory thermal instability in $\gamma=5/3$ plasma may occur ($\alpha<0.8$ as determined in Ref. [4]), shocks are never subject to the dynamical overstability, and overstable shocks always have values of α that render them stable to the linear oscillations. However, $\alpha>0.8$, although a necessary condition for dynamical overstability, is not sufficient. The precise conditions for overstability in terms of γ_{eff}, the Mach number and thickness of the shell of shocked plasma, may be calculated from analytic formulas given in Ref. [8].

III. COMPARISON WITH EXPERIMENTS

While the main virtue of the analytic approach pursued in this Brief Report is in the transparent picture given of the relationship between the two instabilities of radiative shock waves, sufficient detail exists to allow us to compare with the experimental results of Ref. [10], already modeled and further interpreted to a certain extent in Refs. [12,13]. Behind shock waves the magnitude of α will be reduced due to the ionization nonequilibrium, and increased by the nonequilibrium partitioning of energy between ions and electrons. In the case of shocks in Xe, the radiative preheating is so strong that the second effect easily dominates. Writing

$$\Lambda \propto \left(\frac{n_iT_i+n_eT_e}{n_i+n_e}\right)^{\alpha_{noneq}} \propto T_{eq}^{\alpha_{eq}} \tag{5}$$

so that

$$\frac{d\Lambda}{dT_e}=\alpha_{noneq}\frac{\Lambda n_e}{n_iT_i+n_eT_e}=\alpha_{eq}\frac{\Lambda}{T_{eq}}, \tag{6}$$

we derive

$$\alpha_{noneq}=\alpha_{eq}\frac{n_iT_i+n_eT_e}{n_eT_{eq}}. \tag{7}$$

Identifying T_e in nonequilibrium conditions with T_{eq} in equilibrium, $\alpha_{noneq}\simeq\alpha_{eq}(1+n_iT_i/n_eT_e)$. In ionization equilibrium at temperatures approaching 10^6 K ($T_i=T_e$ and $n_i\gg n_e$), Xe

TABLE II. Accreted column density.

α	$\int_0^{x_s} \rho dx / \rho_{in} x_s$
-1	$\left[\frac{\gamma}{3}\left(\frac{\gamma+1}{\gamma-1}\right) - \frac{2\gamma+1}{4} + \left(\frac{\gamma-1}{\gamma+1}\right)\frac{\gamma+1}{5}\right]\left[\frac{\gamma}{4} - \left(\frac{\gamma-1}{\gamma+1}\right)\frac{2\gamma+1}{5} + \left(\frac{\gamma-1}{\gamma+1}\right)^2\frac{\gamma+1}{6}\right]^{-1}$
0	$\left(\frac{\gamma+1}{\gamma-1}\right)\left[\frac{-\gamma-1}{6}\right]\left[\frac{-\gamma-3}{12}\right]^{-1} = 2\left(\frac{\gamma+2}{\gamma+3}\right)\left(\frac{\gamma+1}{\gamma-1}\right)$
$1/2$	$\left(\frac{\gamma+1}{\gamma-1}\right)\left[\frac{\gamma+1}{4}\sqrt{\frac{\gamma-1}{\gamma+1} - \left(\frac{\gamma-1}{\gamma+1}\right)^2} + \frac{\gamma-3}{8}\cos^{-1}\left(1 - 2\frac{\gamma-1}{\gamma+1}\right)\right]\Bigg\{\sqrt{\frac{\gamma+1}{\gamma-1} - 1} \times\left[-\frac{\gamma+1}{3}\left(\frac{\gamma-1}{\gamma+1}\right)^2 - \frac{5-\gamma}{12}\frac{\gamma-1}{\gamma+1} + \frac{\gamma-5}{8} + \frac{5-\gamma}{16}\cos^{-1}\left(1 - 2\frac{\gamma-1}{\gamma+1}\right)\right]\Bigg\}^{-1}$
1	$\left[\gamma - 1 + \ln\frac{2}{\gamma+1}\right]\left[\frac{\gamma-1}{2} + \ln\frac{2}{\gamma+1}\right]^{-1}$
2	$\left[\gamma \ln\left(\frac{\gamma-1}{\gamma+1}\gamma M^2\right) + \frac{1-\gamma}{2}\right]\left[\frac{\gamma-1}{2} + (\gamma+1)\ln\frac{2}{\gamma+1}\right]^{-1}$

obeys a cooling law with $\alpha \simeq 2$, as Λ rises from 10^{-19} ergs cm^{-6} s^{-1} at 10^5 K to 10^{-17} ergs cm^{-6} s^{-1} at 10^6 K [12–14]. Therefore in the shocked nonequilibrium plasma where $n_i T_i \geqslant n_e T_e$ [12,13], we expect $\alpha \simeq 3$ or more. In Fig. 2 we plot the real part of the growth exponent Re(s) and its angular degree l calculated from the analytic theory in Ref. [8] for shocks of Mach numbers 10, 20, and 30, for $\alpha = 3$, which are very similar to those calculated using the detailed atomic physics calculations in Refs. [12,13]. From this calculation and from Table III we estimate a minimum Mach number of 10–20 for a minimum shock compression of the same order, in surprisingly good agreement with the more detailed analyses in Refs. [12,13]. The existence of a minimum shock Mach number for overstability, suggested by the behavior of the overstability in Ref. [10], and bolstered by its absence in the lower velocity shocks studied in Ref. [11], is now much more firmly understood. Given that the shock can only cool down to a temperature similar to its preshock temperature, the amount of cooling and hence shock compression possible will be higher if the shock Mach number is higher.

Similar considerations for N are less straightforward since in the experiments the N cooling rate is more affected by the electron density [13], as electrons collisionally depopulate excited levels. Taking $\Lambda \propto n_e^{\beta} T_e^{\alpha}$ with $1 \leqslant \beta \leqslant 2$ ($\beta = 2$ for the Xe shocks considered above), we should expect the onset of dynamical overstability for $\alpha > \beta$ [6], i.e., at lower α than for

TABLE III. Shock compressions.

α	5/3	7/5	4/3
-1	5.585	8.232	9.560
0	6.286	9.273	10.77
1/2	6.989	10.22	11.82
1	8.302	12.31	14.32
2	$7.684 \ln M - 4.132$	$11.78 \ln M - 9.418$	$13.82 \ln M - 12.32$

Xe. The radiative heating of the shock precursor is also less strong than in Xe, so we expect some further reduction in α due to the increase in electron temperature at the shock front. In the low density limit, the N radiative cooling function declines steeply with temperature between about 10^5 and 10^6 K, giving $\alpha \simeq -3$ [14]. It is difficult to determine its nonequilibrium value, but it seems safe to say that it remains < 0 for most of the blast wave evolution in [10], yielding no overstability as observed.

So far we have only considered "steady state" radiative shocks as an approximation to the decelerating shocks that can become dynamically overstable. The deceleration will of course change the postshock density and allow overstability in a slightly different parameter range. Planar shocks decelerating from v to v' will have their postshock compression increased by approximately v/v'. Spherical shocks change their shock compression by $vr'^2/v'r^2$, where r and r' are initial and final radii. Consequently shocks decelerating faster than $v \propto r^{-2}$ compress further (e.g., the pressure driven snowplow, $v \propto r^{-5/2}$) while those with slower deceleration expand (e.g., Sedov-Taylor $v \propto r^{-3/2}$).

Detailed atomic physics has so far been absent from our discussion. It will of course come into the value of $x_s \simeq u_{in}[\gamma - 1/\gamma + 1] k_B T / n \Lambda T^{\alpha}$ [15], and into the shock Mach number, which is reduced from the nominal value obtained by dividing the shock speed by the initial gas sound speed by the heating and ionization of the shock precursor by UV and x radiation from the shocked gas [12,13]. In considering the dynamical overstability of spherical blast waves, x_s may be considered the thickness of the shocked shell, which if too thick suppresses the overstability. In the extreme case of x_s being larger than the shock dimensions, the shock ceases to be radiative, no matter what the value of α. So long as the shock is radiative, and can be said to be at least approximately in a steady state, one may evaluate the effective polytropic index from Eqs. (1)–(3), input this together with $D = x_s$ and the Mach number into the analytic theory in Ref. [8] to determine the overstability.

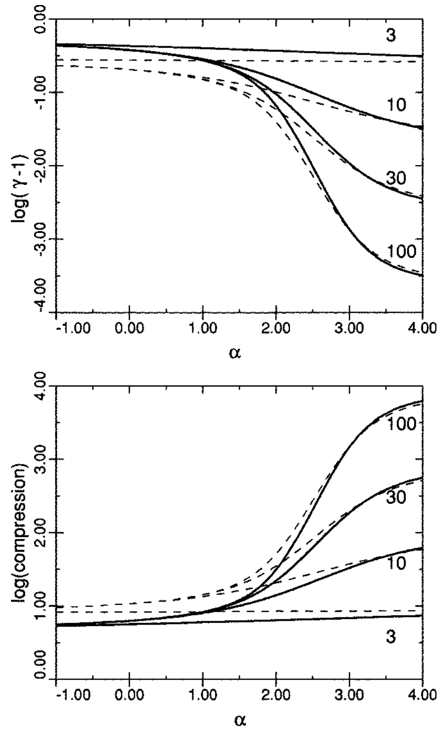

FIG. 1. Plots of shock compression (lower panel) and effective polytropic index (upper panel) against α for shock Mach numbers of 3, 10, 30, and 100 in $\gamma=5/3$ gas (solid line) and in $\gamma=4/3$ gas (dashed line). In regions of α where the oscillatory instability occurs for $\gamma=5/3$, the dynamical overstability cannot grow, since insufficient shock compression takes places. Only for $\alpha \geqslant 2$ can sufficient shock compression occur to drive the overstability, and then only if the Mach number is above a certain critical value. See text for further discussion.

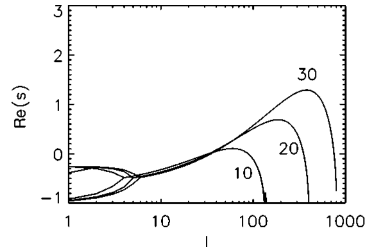

FIG. 2. Overstability growth exponent Re(s) against l for Sedov-Taylor blast waves with Mach numbers of 10, 20, and 30 and $\alpha=3$, computed from the analytic theory in Ref. [8]. The maximum in Re(s) and the value of l at which it occurs increases with increasing Mach number. The values of γ_{eff} are taken from the shock compressions calculated in Fig. 1.

IV. CONCLUSIONS

We have extended the analytic description of radiative shocks given initially in Ref. [4] to model the dynamical overstability of the layer of cooling gas. This is a qualitative departure from earlier works [5–8], who only considered such behavior in the thin shell of already cooled gas, but more consistent with our earlier detailed treatments of the radiative cooling [12,13]. We find, at least in $\gamma=5/3$ gas, that the oscillatory thermal instability modeled in Ref. [4] and the dynamical overstability in the cooling gas are mutually exclusive. No form of the radiative power loss, $\Lambda \propto T^{\alpha}$, can give rise to both instabilities simultaneously. We conjecture that such a relationship between the two forms of instability should exist for lower γ. From Table III it is clear that the dynamical thin-shell overstability can occur for lower values of α in lower γ gas, i.e., becomes more prevalent. In Eq. (3), the term driving the thermal instability $\propto \gamma-1$, so as $\gamma \to 1$, we should expect the oscillatory thermal instability to become less prevalent. This indeed seems to be the case. Instability for the first overtone for $\alpha \leqslant -0.4$ and for the fundamental for $\alpha \leqslant -4$ for $\gamma=21/19$ gas (as opposed to 0.8 and 0.4 for $\gamma=5/3$) are reported in Ref. [16].

We have also shown that the dynamical thin-shell overstability theory applied to the cooling gas gives a good account of the main observations of such instabilities in Ref. [10]. Inspection of the numerical results in Refs. [12,13] reveals that except at the earliest times, this is a good approximation, since the expansion of the cooled gas shell as the blast wave expands reduces it compression, and the overall behavior of the blast wave in increasingly dominated by the more recently shocked gas.

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